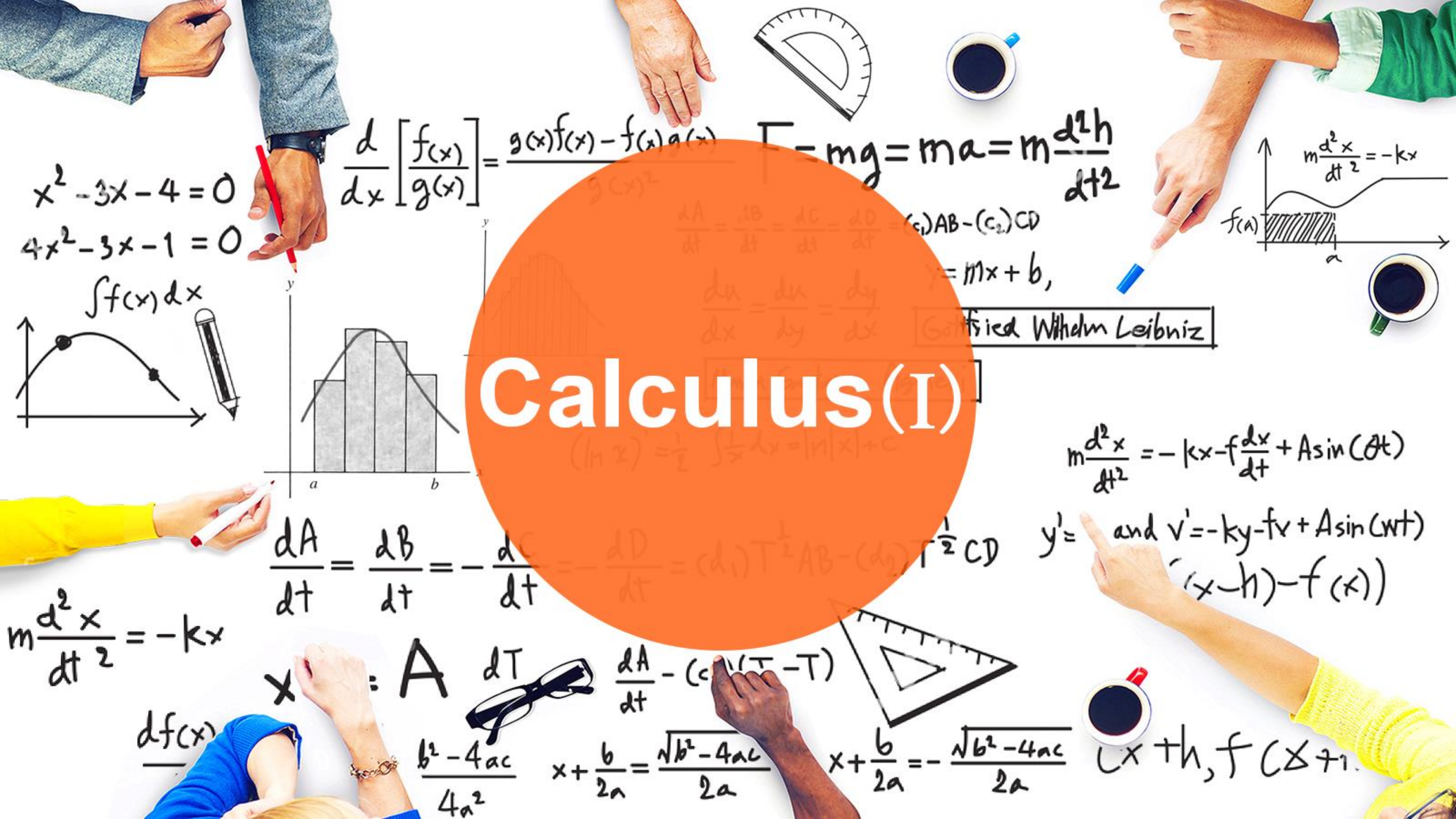
Calculus(Ⅰ)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
m d²x/dt² = -kx
y = mx + b,
m d²x/dt² = -kx - f dx/dt + Asin(ωt)
and v' = -ky - fv + Asin(wt)

2.9 Differentials and Approximations

Lecturer: Xue Deng

Definition of the Differentials

Let $y = f(x)$ be a differentiable function of the independent variable x.

Δx is an arbitrary increment in the independent variable x.

dx, called the **differential of the independent variable** x, is equal to Δx.

Δy is the actual change in the variable y as x changes from x to $x + \Delta x$; that is, $\Delta y = f(x + \Delta x) - f(x)$.

dy, called the **differential of the dependent variable** y, is defined by $dy = f'(x)dx$.

Differential Rule

Solution:

Calculate the derivative and multiply it by the differential of the independent variable.

Namely,

$$dy = f'(x)dx$$

Basic Differential Formula

$$d(C) = 0 \qquad d(x^{\mu}) = \mu x^{\mu-1}dx$$

$$d(\sin x) = \cos x dx \qquad d(\cos x) = -\sin x dx$$

$$d(\tan x) = \sec^2 x dx \qquad d(\cot x) = -\csc^2 x dx$$

$$d(\sec x) = \sec x \tan x dx \qquad d(\csc x) = -\csc x \cot x dx$$

$$d(a^x) = a^x \ln a dx \qquad d(e^x) = e^x dx$$

$$d(\log_a x) = \frac{1}{x \ln a}dx \qquad d(\ln x) = \frac{1}{x}dx$$

$$d(\arcsin x) = \frac{1}{\sqrt{1-x^2}}dx \qquad d(\arccos x) = -\frac{1}{\sqrt{1-x^2}}dx$$

$$d(\arctan x) = \frac{1}{1+x^2}dx \qquad d(\operatorname{arccot} x) = -\frac{1}{1+x^2}dx$$

Differential Rule

Rules:

$$(u = u(x), v = v(x), \alpha, \beta \in R)$$

$$d(\alpha u + \beta v) = \alpha du + \beta dv$$

$$d(uv) = vdu + udv$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

Example 1

Find dy if $y = \ln\left(x + e^{x^2}\right)$.

$$\because\ y' = \frac{1 + 2xe^{x^2}}{x + e^{x^2}},$$

$$\therefore\ \mathrm{d}y = \frac{1 + 2xe^{x^2}}{x + e^{x^2}}\mathrm{d}x.$$

Example 2

$$d(uv) = v\,du + u\,dv$$

Find dy if $y = e^{1-3x}\cos x$.

$$dy = \cos x \cdot d(e^{1-3x}) + e^{1-3x} \cdot d(\cos x)$$

$$\because (e^{1-3x})' = -3e^{1-3x},$$

$$(\cos x)' = -\sin x.$$

$$\therefore dy = \cos x \cdot (-3e^{1-3x})dx + e^{1-3x} \cdot (-\sin x)dx$$

$$= -e^{1-3x}(3\cos x + \sin x)dx.$$

Differential Formula of Composite Function

If $f'(x)$ is the Derivative of function $y = f(x)$,

（1）For the independent variable x,

$dy = f'(x)\,dx;$

（2）For the intermediate variable x,

Namely, $x = \varphi(t)$ is differentiable,

Then, $\mathbf{dy = f'(x) \cdot \varphi'(t)dt}$

dx

$\mathbf{dy = f'(x)dx.}$

Example 3

Find dy if $y = e^{ax+bx^2}$.

$$\mathrm{d}y = \left(e^{ax+bx^2}\right)'\mathrm{d}x = e^{ax+bx^2}\cdot(a+2bx)\mathrm{d}x$$

Example 4

Find dy if $y = \ln^{\mu} x$.

$$\mathrm{d}(\ln^{\mu} x) = \mu \ln^{\mu-1} x \,\mathrm{d}\,(\ln x)$$

$$= \frac{\mu}{x} \ln^{\mu-1} x \mathrm{d}x$$

Example 5

Find $d(x \tan^{-1} 2x)$.

$$d(x \arctan 2x)$$

$$= \arctan 2x dx + x \cdot d(\arctan 2x)$$

$$= \arctan 2x dx + x \cdot \frac{1}{1+(2x)^2} d(2x)$$

$$= [\arctan 2x + \frac{2x}{1+(2x)^2}] dx$$

Example 6

Fill in the blanks by proper functions, so that Equations hold.

(1) $d(\quad) = \cos\omega t\, dt$; (2) $d(\sin x^2) = (\quad) d(\sqrt{x})$.

(1) $\mathrm{d}(\sin\omega t) = \omega\cos\omega t\mathrm{d}t$,

$\therefore \cos\omega t\mathrm{d}t = \frac{1}{\omega}\mathrm{d}(\sin\omega t) = \mathrm{d}(\frac{1}{\omega}\sin\omega t)$;

$\therefore \mathrm{d}(\frac{1}{\omega}\sin\omega t + C) = \cos\omega t\mathrm{d}t$.

(2) $\frac{\mathrm{d}(\sin x^2)}{\mathrm{d}(\sqrt{x})} = \frac{2x\cos x^2\mathrm{d}x}{\frac{1}{2\sqrt{x}}\mathrm{d}x} = 4x\sqrt{x}\cos x^2$,

$\therefore \mathrm{d}(\sin x^2) = (4x\sqrt{x}\cos x^2)\mathrm{d}(\sqrt{x})$.

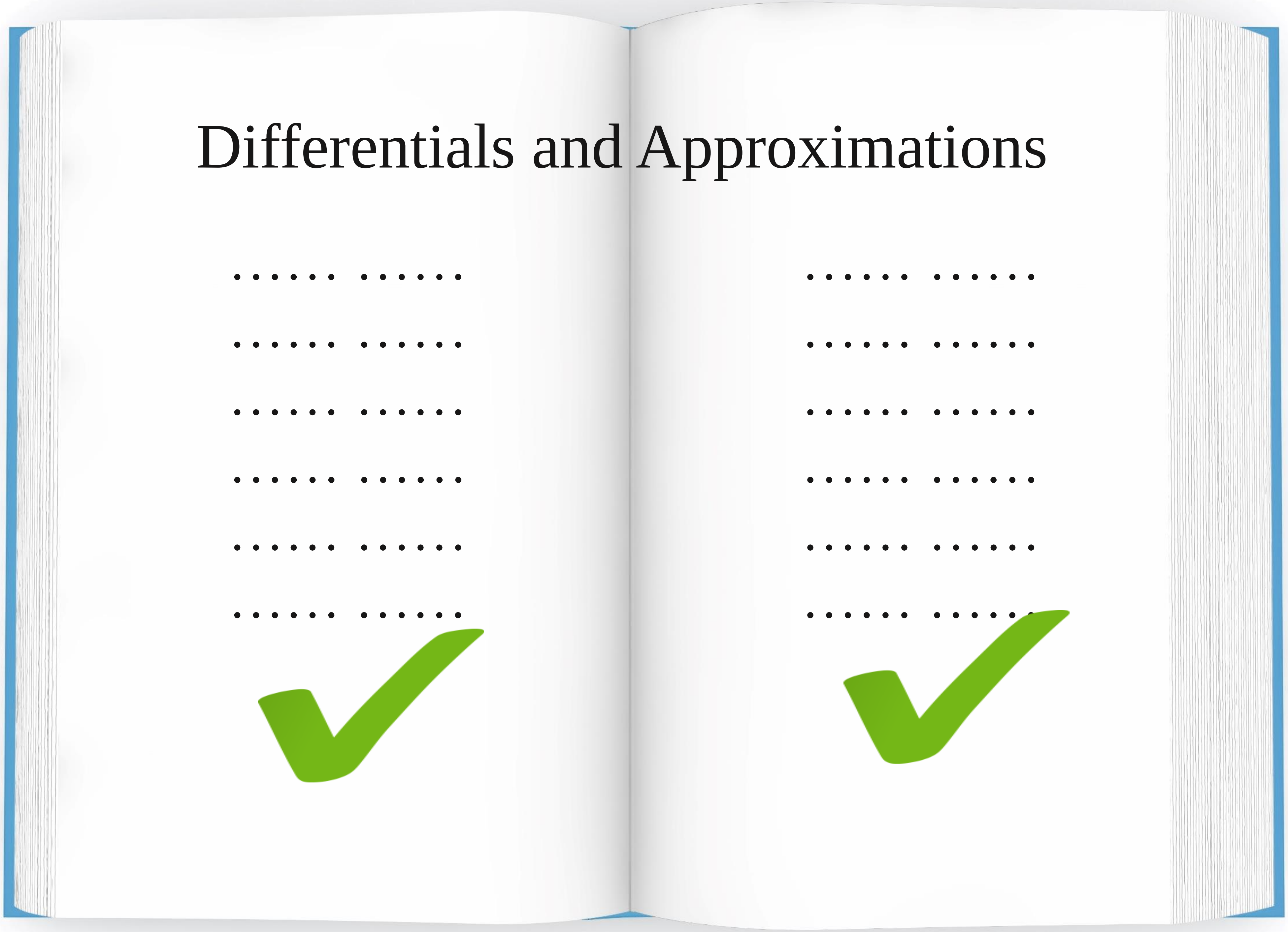
Example 7

Find dy if $x^2y + xy^2 = 1$.

$$\mathrm{d}(x^2y + xy^2) = 0$$

$$x^2\mathrm{d}y + y2x\mathrm{d}x + y^2\mathrm{d}x + 2xy\mathrm{d}y = \mathbf{0}$$

$$\mathrm{d}y = -\frac{2xy + y^2}{x^2 + 2xy}\mathrm{d}x.$$


Differentials and Approximations